Chapter 6. Radiating Systems

Notes:

- *Most of the material presented in this chapter is taken from Jackson, Chap. 9, and Rybicki and Lightman, Chap. 3.*

6.1 Radiation from a Localized Oscillating Source

Since any function, say $\rho(\mathbf{x},t)$ of position and time can always be expressed with Fourier transform with

$$\begin{aligned}\rho(\mathbf{x},t) &= \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\rho(\mathbf{k},\omega)e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)}\,d^3k\,d\omega \\ \rho(\mathbf{k},\omega) &= \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\rho(\mathbf{x},t)e^{-i(\mathbf{k}\cdot\mathbf{x}-\omega t)}\,d^3x\,dt,\end{aligned} \tag{6.1}$$

it will always be possible to integrate on only one of the subspaces in any of equations (6.1) to get for example

$$\begin{aligned}\rho(\mathbf{x},t) &= \frac{1}{4\pi^2}\int_{-\infty}^{\infty}d\omega e^{-i\omega t}\int_{-\infty}^{\infty}\rho(\mathbf{k},\omega)e^{i\mathbf{k}\cdot\mathbf{x}}\,d^3k \\ &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\rho(\mathbf{x},\omega)e^{-i\omega t}\,d\omega,\end{aligned} \tag{6.2}$$

with

$$\rho(\mathbf{x},\omega) = \frac{1}{(2\pi)^{3/2}}\int_{-\infty}^{\infty}\rho(\mathbf{k},\omega)e^{i\mathbf{k}\cdot\mathbf{x}}\,d^3k. \tag{6.3}$$

It is, therefore, sensible to simplify problems by considering only one particular frequency component at a time (in a way similar as was previously done). For example, for the sources of potentials and electromagnetic fields (i.e., the charge and current densities) we can write, with a slight modification to the notation used in equation (6.2),

$$\begin{aligned}\rho(\mathbf{x},t) &= \rho(\mathbf{x})e^{-i\omega t} \\ \mathbf{J}(\mathbf{x},t) &= \mathbf{J}(\mathbf{x})e^{-i\omega t}.\end{aligned} \tag{6.4}$$

The (frequency components of the) potentials and sources will exhibit the same time dependency, and as usual, the corresponding physical quantities are obtained by taking the real part of expressions such as equations (6.4). For the present purposes, we will assume that the sources are localized in free space. In fact, we will consider situations where the size d of the sources is much smaller that the wavelength, with $d \ll \lambda$.

As a first step, we can use the current density of equation (6.4) to evaluate the fields from the last of equations (4.38) for the vector potential. So, we write (using the Lorenz gauge)

$$\begin{aligned}\mathbf{A}(\mathbf{x},t) &= \frac{\mu_0}{4\pi}\int \frac{\left[\mathbf{J}(\mathbf{x}',t')\right]_{\text{ret}}}{|\mathbf{x}-\mathbf{x}'|}\,d^3x' \\ &= \frac{\mu_0}{4\pi}\int d^3x' \int dt' \frac{\mathbf{J}(\mathbf{x}',t')}{|\mathbf{x}-\mathbf{x}'|}\delta\left(t'-t+\frac{|\mathbf{x}-\mathbf{x}'|}{c}\right).\end{aligned} \tag{6.5}$$

If we insert the second of equations (6.4) in equation (6.5) we find that

$$\begin{aligned}\mathbf{A}(\mathbf{x},t) &= \frac{\mu_0}{4\pi}\int d^3x' \frac{\mathbf{J}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}\int dt'\, e^{-i\omega t'}\delta\left(t'-t+\frac{|\mathbf{x}-\mathbf{x}'|}{c}\right) \\ &= e^{-i\omega t}\frac{\mu_0}{4\pi}\int \mathbf{J}(\mathbf{x}')\frac{e^{ik|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|}\,d^3x',\end{aligned} \tag{6.6}$$

with $k = \omega/c$, and

$$\boxed{\mathbf{A}(\mathbf{x}) = \frac{\mu_0}{4\pi}\int \mathbf{J}(\mathbf{x}')\frac{e^{ik|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|}\,d^3x'} \tag{6.7}$$

The electromagnetic fields in free space are then easily obtained from this result and

$$\mathbf{H} = \frac{1}{\mu_0}\nabla\times\mathbf{A}, \tag{6.8}$$

and the Ampère-Maxwell law (with $\mathbf{J} = 0$ in free space)

$$\mathbf{E} = i\frac{Z_0}{k}\nabla\times\mathbf{H}, \tag{6.9}$$

with $Z_0 = \sqrt{\mu_0/\varepsilon_0}$ the impedance of vacuum.

Because it may not be possible to solve the integral in equation (6.7) for a general current distribution, we want to work a simple approximation procedure that will allow us to evaluate the dominant terms for the vector potential. More precisely, in the so-called **near field** where $d \ll r \ll \lambda$ the exponential in the integrand of equation (6.7) approximately equals one and we can write, substituting equation (2.61) for $|\mathbf{x}-\mathbf{x}'|^{-1}$,

$$\lim_{kr\to 0}\mathbf{A}(\mathbf{x}) = \frac{\mu_0}{4\pi}\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{4\pi}{2l+1}\frac{Y_{lm}(\theta,\varphi)}{r^{l+1}}\int \mathbf{J}(\mathbf{x}')r'^{l}Y_{lm}^{*}(\theta',\varphi')\,d^3x', \tag{6.10}$$

where $r = |\mathbf{x}|$ and $r' = |\mathbf{x}'|$. Equation (6.10) shows that, although it exhibits sinusoidal oscillations, the vector potential is otherwise static in form (that is, in a spatial sense). On the other hand, in the **far field** where $d \ll \lambda \ll r$, the exponential in equation (6.7) oscillates rapidly, and therefore dictates the behavior of the potential and we make the following approximation

$$|\mathbf{x} - \mathbf{x}'| \simeq r - \mathbf{n} \cdot \mathbf{x}', \tag{6.11}$$

with $\mathbf{n} = \mathbf{e}_r$ is the unit vector directed from the source to the observation point (actually, equation (6.11) is good even in the near field since $d \ll r$ there). However, it is sufficient to keep only the leading term on the right-hand side of equation (6.11) for the denominator, which, when combined with the approximation for the exponential, yields for the vector potential

$$\lim_{kr \to \infty} \mathbf{A}(\mathbf{x}) = \frac{\mu_0}{4\pi} \frac{e^{ikr}}{r} \int \mathbf{J}(\mathbf{x}') e^{-ik\mathbf{n}\cdot\mathbf{x}'} \, d^3x'. \tag{6.12}$$

This result implies that the vector potential has only a radial dependency, and propagates as a spherical wave in the r-positive direction. Upon applying the equation for the curl in spherical coordinates we find that the electromagnetic fields evaluated with equations (6.8) and (6.9) are transverse to the radius vector with their amplitude decreasing as $1/r$. They constitute radiation fields.

6.2 The Electric Monopole and Dipole Terms

We have so far only considered expansions of the vector potential. One may inquire as to using similar expansions using the scalar potential instead. To do so, we write the first of equations (4.38) in way similar to what we did in equation (6.5) for the vector potential. That is,

$$\Phi(\mathbf{x},t) = \frac{1}{4\pi\varepsilon_0} \int d^3x' \int dt' \frac{\rho(\mathbf{x}',t')}{|\mathbf{x} - \mathbf{x}'|} \delta\left(t' - t + \frac{|\mathbf{x} - \mathbf{x}'|}{c} \right). \tag{6.13}$$

We can start by considering the electric monopole term, which arises from equation (6.13) when $|\mathbf{x} - \mathbf{x}'| \to |\mathbf{x}| = r$. Then,

$$\begin{aligned} \Phi_{\text{monopole}}(\mathbf{x},t) &= \frac{1}{4\pi\varepsilon_0 r} \int d^3x' \int dt' \rho(\mathbf{x}',t') \delta\left(t' - t + \frac{r}{c} \right) \\ &= \frac{1}{4\pi\varepsilon_0 r} \int \rho\left(\mathbf{x}', t - \frac{r}{c} \right) d^3x' \\ &= \frac{q\left(t' = t - \frac{r}{c} \right)}{4\pi\varepsilon_0 r}. \end{aligned} \tag{6.14}$$

We find that the electric monopole time is proportional to the total charge, and is therefore not a function of time. Since this quantity is conserved, *the fields due to the monopole term are static in character and will* ***not*** *radiate (or dominate) at large distances*.

We now go back to equation (6.12) for the far field approximation of the vector potential, and expand the exponential in its Taylor series about $\mathbf{x}' = 0$. The first term yields the following vector potential

$$\mathbf{A}(\mathbf{x}) = \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\int \mathbf{J}(\mathbf{x}')\, d^3x'. \tag{6.15}$$

We can transform this relation using equation (3.35) derived while dealing with magnetostatics

$$\int \left[g\mathbf{J}\cdot\nabla' f + f\mathbf{J}\cdot\nabla' g + fg\nabla'\cdot\mathbf{J} \right] d^3x' = 0, \tag{6.16}$$

with f and g two good functions. As we did then, we set $f = 1$ and $g = x_i'$ into equation (6.16) to find that

$$\int J_i \; d^3x' = -\int x_i' \nabla'\cdot\mathbf{J}\; d^3x', \tag{6.17}$$

or alternatively

$$\int \mathbf{J}\; d^3x' = -\int \mathbf{x}'\left(\nabla'\cdot\mathbf{J}\right) d^3x'. \tag{6.18}$$

But since from the continuity equation

$$\nabla\cdot\mathbf{J} = i\omega\rho, \tag{6.19}$$

then equation (6.15) becomes

$$\mathbf{A}(\mathbf{x}) = -\frac{\mu_0 i\omega}{4\pi}\frac{e^{ikr}}{r}\int \mathbf{x}'\rho(\mathbf{x}')\, d^3x'. \tag{6.20}$$

Just as we did in electrostatics, we define the **electric dipole moment** as

$$\boxed{\mathbf{p} = \int \mathbf{x}'\rho(\mathbf{x}')\, d^3x'} \tag{6.21}$$

and

$$\boxed{\mathbf{A}(\mathbf{x}) = -\frac{\mu_0 i\omega}{4\pi}\mathbf{p}\frac{e^{ikr}}{r}} \tag{6.22}$$

The electromagnetic fields can be evaluated from equations (6.8) and (6.9) with

$$\begin{aligned}
\mathbf{H} &= -\frac{i\omega}{4\pi}\nabla\times\left(\mathbf{p}\frac{e^{ikr}}{r}\right) \\
&= \frac{i\omega}{4\pi}\mathbf{p}\times\nabla\left(\frac{e^{ikr}}{r}\right) \\
&= \frac{ck^2}{4\pi}(\mathbf{n}\times\mathbf{p})\frac{e^{ikr}}{r}\left(1-\frac{1}{ikr}\right),
\end{aligned} \tag{6.23}$$

and

$$\begin{aligned}
\mathbf{E} &= i\frac{Z_0}{k}\nabla\times\left[\frac{ck^2}{4\pi}(\mathbf{n}\times\mathbf{p})\frac{e^{ikr}}{r}\left(1-\frac{1}{ikr}\right)\right] \\
&= \frac{ik}{4\pi\varepsilon_0}\nabla\times\left[(\mathbf{n}\times\mathbf{p})\frac{e^{ikr}}{r}\left(1-\frac{1}{ikr}\right)\right] \\
&= \frac{ik}{4\pi\varepsilon_0}\left\{\nabla\left[\frac{e^{ikr}}{r}\left(1-\frac{1}{ikr}\right)\right]\times(\mathbf{n}\times\mathbf{p}) + \frac{e^{ikr}}{r}\left(1-\frac{1}{ikr}\right)\nabla\times(\mathbf{n}\times\mathbf{p})\right\} \\
&= \frac{ik}{4\pi\varepsilon_0}\left\{\left[ik\frac{e^{ikr}}{r}\left(1-\frac{1}{ikr}\right)^2 + \frac{e^{ikr}}{ikr^3}\right]\mathbf{n}\times(\mathbf{n}\times\mathbf{p}) + \frac{e^{ikr}}{r}\left(1-\frac{1}{ikr}\right)\left[(\mathbf{p}\cdot\nabla)\mathbf{n} - \mathbf{p}(\nabla\cdot\mathbf{n})\right]\right\} \\
&= \frac{ik}{4\pi\varepsilon_0}\left\{\left[\frac{e^{ikr}}{r}\left(ik - \frac{2}{r} + \frac{2}{ikr^2}\right)\right]\mathbf{n}\times(\mathbf{n}\times\mathbf{p}) + \frac{e^{ikr}}{r}\left(\frac{1}{r}-\frac{1}{ikr^2}\right)\left[\left(p_\theta\mathbf{e}_\theta + p_\varphi\mathbf{e}_\varphi\right) - 2\mathbf{p}\right]\right\} \\
&= \frac{1}{4\pi\varepsilon_0}\left\{(\mathbf{n}\times\mathbf{p})\times\mathbf{n}\,k^2\frac{e^{ikr}}{r} - \frac{2e^{ikr}}{r}\left(\frac{ik}{r}-\frac{1}{r^2}\right)\left[\mathbf{n}(\mathbf{n}\cdot\mathbf{p}) - \mathbf{p}\right]\right. \\
&\quad \left. -\frac{e^{ikr}}{r}\left(\frac{ik}{r}-\frac{1}{r^2}\right)\left[\mathbf{n}(\mathbf{n}\cdot\mathbf{p}) + \mathbf{p}\right]\right\} \\
&= \frac{1}{4\pi\varepsilon_0}\left\{(\mathbf{n}\times\mathbf{p})\times\mathbf{n}\,k^2\frac{e^{ikr}}{r} + \left[3\mathbf{n}(\mathbf{n}\cdot\mathbf{p}) - \mathbf{p}\right]\left(\frac{1}{r^3}-\frac{ik}{r^2}\right)e^{ikr}\right\}.
\end{aligned} \tag{6.24}$$

In the near field (i.e., $kr \to 0$), equations (6.23) and (6.24) simplify to

$$\boxed{\begin{aligned}\mathbf{H} &= \frac{i\omega}{4\pi}(\mathbf{n}\times\mathbf{p})\frac{1}{r^2}\\ \mathbf{E} &= \frac{1}{4\pi\varepsilon_0}\left[3\mathbf{n}(\mathbf{n}\cdot\mathbf{p})-\mathbf{p}\right]\frac{1}{r^3}\end{aligned}} \tag{6.25}$$

Comparison with the corresponding electrostatics relation (i.e., equation (2.97)) shows that, except for its implicit time oscillations (see equations (6.4)), the electric dipole field in the near field is similar to its static counterpart. Since, from equation (6.9), the magnetic field times Z_0 is smaller than the electric field by a factor proportional to kr, then the near field is dominantly electric in nature.

In the far field (i.e., $kr \to \infty$), the electromagnetic fields are

$$\boxed{\begin{aligned}\mathbf{H} &= \frac{ck^2}{4\pi}(\mathbf{n}\times\mathbf{p})\frac{e^{ikr}}{r}\\ \mathbf{E} &= \frac{1}{4\pi\varepsilon_0}\left[(\mathbf{n}\times\mathbf{p})\times\mathbf{n}\right]k^2\frac{e^{ikr}}{r}\end{aligned}} \tag{6.26}$$

It is important to note that since $k = \omega/c$, it is possible to show (see the third problem list) that equations (6.26) can be rewritten as

$$\boxed{\begin{aligned}\mathbf{H}(\mathbf{x},t) &= \frac{1}{4\pi cr}\left(\frac{\partial^2\mathbf{p}_{\text{ret}}}{\partial t^2}\times\mathbf{n}\right)\\ \mathbf{E}(\mathbf{x},t) &= \frac{1}{4\pi\varepsilon_0 c^2 r}\left[\mathbf{n}\times\left(\mathbf{n}\times\frac{\partial^2\mathbf{p}_{\text{ret}}}{\partial t^2}\right)\right]\end{aligned}} \tag{6.27}$$

where $\mathbf{p}_{\text{ret}} = \mathbf{p}(t' = t - r/c)$ (i.e., $\mathbf{p}_{\text{ret}}$ is the dipole moment evaluated at the retarded time). We, therefore, once again (see equations (4.62)) obtain the fundamental result that *radiation fields are due to the acceleration of charges*. More precisely, we see here that *the dominant radiation fields come from the acceleration of the electric dipole moment*.

The time-averaged power radiated per unit solid angle by the oscillating dipole moment is given by

$$\frac{dP}{d\Omega} = \frac{1}{2}r^2\mathbf{n}\cdot\left(\mathbf{E}\times\mathbf{H}^*\right), \tag{6.28}$$

and substituting equations (6.26) we find

$$\begin{aligned}\frac{dP}{d\Omega} &= \frac{ck^4}{32\pi^2\varepsilon_0}\left(\mathbf{n}\cdot\left\{\left[\left(\mathbf{n}\times\mathbf{p}\right)\times\mathbf{n}\right]\times\left(\mathbf{n}\times\mathbf{p}\right)^*\right\}\right) \\ &= \frac{ck^4}{32\pi^2\varepsilon_0}\left\{\left[\left(\mathbf{n}\times\mathbf{p}\right)\times\mathbf{n}\right]\cdot\left[\left(\mathbf{n}\times\mathbf{p}\right)\times\mathbf{n}\right]^*\right\},\end{aligned} \tag{6.29}$$

where we used the relation $\mathbf{a}\cdot\left(\mathbf{b}\times\mathbf{c}\right)=\mathbf{b}\cdot\left(\mathbf{c}\times\mathbf{a}\right)$. Then,

$$\boxed{\frac{dP}{d\Omega} = \frac{ck^4}{32\pi^2\varepsilon_0}\left|\left(\mathbf{n}\times\mathbf{p}\right)\times\mathbf{n}\right|^2} \tag{6.30}$$

or alternatively

$$\boxed{\frac{dP}{d\Omega} = \frac{Z_0c^2k^4}{32\pi^2}\left|\left(\mathbf{n}\times\mathbf{p}\right)\times\mathbf{n}\right|^2} \tag{6.31}$$

The state of polarization of the radiation is given by $\left(\mathbf{n}\times\mathbf{p}\right)\times\mathbf{n}$. If the phase of the different component of the dipole moment is the same, then the angular distribution is that of a typical dipole pattern

$$\frac{dP}{d\Omega} = \frac{Z_0c^2k^4}{32\pi^2}\left|\mathbf{p}\right|^2\sin^2\left(\theta\right), \tag{6.32}$$

where θ is the angle measured from $\mathbf{p}$ to $\mathbf{n}$. Finally, the total power irradiated, irrespective of the relative phases of the components of the dipole moment is

$$\begin{aligned}P &= \frac{Z_0c^2k^4}{32\pi^2}\left|\mathbf{p}\right|^2\int\sin^2\left(\theta\right)\,d\Omega \\ &= \frac{Z_0c^2k^4}{16\pi}\left|\mathbf{p}\right|^2\int_0^\pi\sin^3\left(\theta\right)\,d\theta \\ &= \frac{Z_0c^2k^4}{16\pi}\left|\mathbf{p}\right|^2\int_0^\pi\sin\left(\theta\right)\left[1-\cos^2\left(\theta\right)\right]\,d\theta,\end{aligned} \tag{6.33}$$

which yields

$$\boxed{P = \frac{Z_0c^2k^4}{12\pi}\left|\mathbf{p}\right|^2} \tag{6.34}$$

6.2.1 **An Example: Thomson Scattering of Radiation**

One important application of the dipole approximation is to the process in which a free charge radiates in response to an incident electromagnetic monochromatic wave. If the charge oscillates at non-relativistic velocities ($v \ll c$), then we can neglect the magnetic term of the Lorentz force (since $E = Z_0 H = cB$ for a transverse wave in vacuum). If the electric field associated with the wave is

$$\mathbf{E}(\mathbf{x}) = \mathbf{e}_0 E_0 e^{i\mathbf{k}\cdot\mathbf{x}}, \tag{6.35}$$

(again, this refers to the temporal Fourier transform of the field; see equations (6.4)), then considering the Lorentz force on the charge we have

$$\mathbf{x} = -\mathbf{e}_0 \frac{q}{\omega^2 m} E_0 e^{i\mathbf{k}\cdot\mathbf{x}}. \tag{6.36}$$

The dipole moment $\mathbf{p} = q\mathbf{x}$ is therefore

$$\mathbf{p} = -\frac{q^2 E_0}{m\omega^2} e^{i\mathbf{k}\cdot\mathbf{x}} \mathbf{e}_0, \tag{6.37}$$

From equation (6.31), the time-averaged power per unit solid angle polarized along a given direction $\mathbf{e}_1$ perpendicular to the direction of propagation $\mathbf{n}$ will be

$$\begin{aligned} \left(\frac{dP}{d\Omega}\right)_{\text{polarized}} &= \frac{ck^4}{32\pi^2\varepsilon_0}\left|\mathbf{e}_1^* \cdot \left[(\mathbf{n}\times\mathbf{p})\times\mathbf{n}\right]\right|^2 \\ &= \frac{ck^4}{32\pi^2\varepsilon_0}\left|\mathbf{e}_1^* \cdot \left[\mathbf{p} - (\mathbf{n}\cdot\mathbf{p})\mathbf{n}\right]\right|^2, \end{aligned} \tag{6.38}$$

and since $\mathbf{e}_1^* \cdot \mathbf{n} = 0$

$$\boxed{\left(\frac{dP}{d\Omega}\right)_{\text{polarized}} = \frac{ck^4}{32\pi^2\varepsilon_0}\left|\mathbf{e}_1^* \cdot \mathbf{p}\right|^2} \tag{6.39}$$

This result is general. If the process is interpreted as one of scattering, then it is convenient to define the **differential scattering cross section** as

$$\frac{d\sigma}{d\Omega} = \frac{\text{Energy radiated/unit time/unit solid angle}}{\text{Incident energy flux in energy/unit area/unit time}}. \tag{6.40}$$

Mathematically, this translates to

$$\boxed{\frac{dP}{d\Omega} = |\mathbf{S}|\frac{d\sigma}{d\Omega}} \tag{6.41}$$

Since

$$\begin{aligned} |\mathbf{S}| &= \frac{1}{2}\left|\mathbf{E}\times\mathbf{H}^*\right| \\ &= \frac{|\mathbf{E}|^2}{2Z_0} = \frac{1}{2}c\varepsilon_0 |\mathbf{E}|^2, \end{aligned} \tag{6.42}$$

then from equations (6.37) and (6.39)

$$\frac{d\sigma}{d\Omega} = \left(\frac{1}{4\pi\varepsilon_0}\frac{q^2}{mc^2}\right)^2 \left|\mathbf{e}_1^* \cdot \mathbf{e}_0\right|^2. \tag{6.43}$$

When substituting for the mass and charge of an electron, the quantity

$$r_0 \equiv \frac{1}{4\pi\varepsilon_0}\frac{q^2}{mc^2} \tag{6.44}$$

is a measure of the size of an electron ($r_0 = 2.82\times 10^{-15}\,\mathrm{m}$), and is called the **classical electron radius**. Integration of the differential scattering cross section over all solid angle (through a process similar to equations (6.33) yields the **Thomson cross section**

$$\sigma_\mathrm{T} = \frac{8\pi}{3}\left(\frac{1}{4\pi\varepsilon_0}\frac{q^2}{mc^2}\right)^2, \tag{6.45}$$

which equals $0.665\times 10^{-28}\,\mathrm{m}^2$.

6.3 Magnetic Dipole and Electric Quadrupole Fields

We simply use the next term in the expansion of the exponential function in the integrand of equation (6.12) to find expressions for the electromagnetic fields due to the next multipoles in the far field. But in order to find expressions that are valid at any distances from the sources, we will go back to equation (6.7) and substitute the equation (6.11) to approximate both the exponential function, and the denominator. That is,

$$
\begin{aligned}
\frac{e^{ik|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|} &\simeq \frac{e^{ikr}\left(1-ik\mathbf{n}\cdot\mathbf{x}'\right)}{r-\mathbf{n}\cdot\mathbf{x}'} \\
&\simeq \frac{e^{ikr}}{r}\left(1-ik\mathbf{n}\cdot\mathbf{x}'\right)\left(1+\mathbf{n}\cdot\frac{\mathbf{x}'}{r}\right) \\
&\simeq \frac{e^{ikr}}{r}\left[1+\mathbf{n}\cdot\mathbf{x}'\left(\frac{1}{r}-ik\right)\right].
\end{aligned}
\tag{6.46}
$$

The first term on the right-hand side has already been dealt with, and yielded the electric dipole field when inserted in equation (6.7). We are now concerned with the last term on the right-hand side of equation (6.46), with which the potential vector becomes

$$
\mathbf{A}(\mathbf{x}) = \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\left(\frac{1}{r}-ik\right)\int \mathbf{J}(\mathbf{x}')(\mathbf{n}\cdot\mathbf{x}')\, d^3x'. \tag{6.47}
$$

Next, we rearrange the integrand by separating it into parts that are, respectively, symmetric and anti-symmetric in the exchange of $\mathbf{x}'$ and $\mathbf{J}$

$$
\mathbf{n}\times(\mathbf{x}'\times\mathbf{J}) = (\mathbf{n}\cdot\mathbf{J})\mathbf{x}' - (\mathbf{n}\cdot\mathbf{x}')\mathbf{J}, \tag{6.48}
$$

and

$$
(\mathbf{n}\cdot\mathbf{x}')\mathbf{J} = \frac{1}{2}\left[(\mathbf{n}\cdot\mathbf{x}')\mathbf{J} + (\mathbf{n}\cdot\mathbf{J})\mathbf{x}'\right] + \frac{1}{2}(\mathbf{x}'\times\mathbf{J})\times\mathbf{n}. \tag{6.49}
$$

6.3.1 The Magnetic Dipole Fields

The last term on the right-hand side is related to the magnetization due to the current density (see equation (3.39)) with

$$
\boldsymbol{\mathcal{M}}(\mathbf{x}) = \frac{1}{2}(\mathbf{x}\times\mathbf{J}), \tag{6.50}
$$

and the vector potential due to the magnetization is

$$
\boxed{\mathbf{A}(\mathbf{x}) = \frac{\mu_0 ik}{4\pi}(\mathbf{n}\times\mathbf{m})\frac{e^{ikr}}{r}\left(1-\frac{1}{ikr}\right)} \tag{6.51}
$$

with $\mathbf{m}$ the **magnetic dipole moment**

$$
\boxed{\mathbf{m} = \int \boldsymbol{\mathcal{M}}(\mathbf{x}')\, d^3x' = \frac{1}{2}\int (\mathbf{x}'\times\mathbf{J})\, d^3x'} \tag{6.52}
$$

Comparing equation (6.51) with that of the magnetic field $\mathbf{H}_e$ due to the dipole electric moment (i.e., equation (6.23)), we find that they have the same form (more precisely, $\mathbf{A} \to i\mathbf{H}_e \mu_0 / k$, with $\mathbf{p} \to \mathbf{m}/c$). Therefore, since in free space the magnetic field is simply μ_0^{-1} times the curl of the potential vector, and that the electric field is iZ_0/k times the curl of the magnetic field (see equations (6.8) and (6.9)), then the magnetic field due to the magnetic dipole moment will be $\mathbf{H} \to \mathbf{E}_e / Z_0$, with $\mathbf{p} \to \mathbf{m}/c$; $\mathbf{E}_e$ is the electric field dues to the electric dipole moment. That is,

$$\boxed{\mathbf{H} = \frac{1}{4\pi}\left\{k^2 (\mathbf{n} \times \mathbf{m}) \times \mathbf{n} \frac{e^{ikr}}{r} + \left[3\mathbf{n}(\mathbf{n} \cdot \mathbf{m}) - \mathbf{m}\right]\left(\frac{1}{r^3} - \frac{ik}{r^2}\right) e^{ikr}\right\}.} \tag{6.53}$$

In the same way, since from Faraday's law $\mathbf{H} = (\nabla \times \mathbf{E}) / iZ_0 k$, then we find that $\mathbf{E} = ikZ_0 / \mu_0 \mathbf{A}$. That is,

$$\boxed{\mathbf{E}(\mathbf{x}) = -\frac{Z_0}{4\pi} k^2 (\mathbf{n} \times \mathbf{m}) \frac{e^{ikr}}{r}\left(1 - \frac{1}{ikr}\right).} \tag{6.54}$$

Because of the close relationship between the electromagnetic fields due to the electric and magnetic dipole moments, the same comments concerning the near and far fields made in section 6.2 apply here also (that is, as long as we make the following substitutions $\mathbf{E} \to Z_0\mathbf{H}$, $Z_0\mathbf{H} \to -\mathbf{E}$, and $\mathbf{p} \to \mathbf{m}/c$). The power radiated has also the same form in both cases, while the polarization of the wave will be different from the fact that the electric field is oriented along $\mathbf{n} \times \mathbf{m}$ for magnetic dipole radiation (i.e., in a plane perpendicular to $\mathbf{n}$ and $\mathbf{m}$), while is oriented along $(\mathbf{n} \times \mathbf{p}) \times \mathbf{n}$ for electric dipole radiation (i.e., in the plane formed by $\mathbf{n}$ and $\mathbf{p}$).

6.3.2 The Electric Quadrupole Fields

With a little work, the symmetric term of equation (6.49) can be transformed in a way that will allow us to introduce the electric quadrupole term. To do so, we consider a single component, say component "1", of the vector integral

$$\int \left[(\mathbf{n} \cdot \mathbf{x}')\mathbf{J} + (\mathbf{n} \cdot \mathbf{J})\mathbf{x}'\right]_1 d^3x' = \sum_j \int \left[n_j x'_j J_1 + n_j J_j x'_1\right] d^3x', \tag{6.55}$$

where summations on any indices is explicitly written. Next we integrate by parts the terms involving J_1 to get

$$
\begin{aligned}
I &= \sum_j \int \left[n_j x_j' J_1 + n_j J_j x_1' \right] d^3x' \\
&= -\int \left[x_1' \partial_1' J_1 \sum_j n_j x_j' - x_1' \left(n_2 J_2 + n_3 J_3 \right) \right] d^3x',
\end{aligned} \tag{6.56}
$$

and in turn we now integrate by parts the last two terms

$$
\begin{aligned}
I &= -\int x_1' \left[\partial_1' J_1 \sum_j n_j x_j' + n_2 x_2' \partial_2' J_2 + n_3 x_3' \partial_3' J_3 \right] d^3x' \\
&= -\int x_1' \left[(\mathbf{n} \cdot \mathbf{x}')(\nabla' \cdot \mathbf{J}) - \left(n_1 x_1' + n_3 x_3' \right) \partial_2' J_2 \right. \\
&\quad \left. - \left(n_1 x_1' + n_2 x_2' \right) \partial_3' J_3 \right] d^3x' \\
&= -\int x_1' (\mathbf{n} \cdot \mathbf{x}')(\nabla' \cdot \mathbf{J})\, d^3x' + \int \left[x_1' \left(n_1 x_1' + n_3 x_3' \right) J_2 \Big|_{x_2' = -\infty}^{x_2' = \infty} \right] dx_1' dx_3' \\
&\quad + \int \left[x_1' \left(n_1 x_1' + n_2 x_2' \right) J_3 \Big|_{x_3' = -\infty}^{x_3' = \infty} \right] dx_1' dx_2' \\
&= -\int x_1' (\mathbf{n} \cdot \mathbf{x}')(\nabla' \cdot \mathbf{J})\, d^3x'.
\end{aligned} \tag{6.57}
$$

Finally, resorting back to a vector notation, and using the equation of continuity (see equation (6.19)), we have

$$
\frac{1}{2} \int \left[(\mathbf{n} \cdot \mathbf{x}') \mathbf{J} + (\mathbf{n} \cdot \mathbf{J}) \mathbf{x}' \right] d^3x' = -\frac{i\omega}{2} \int \mathbf{x}' (\mathbf{n} \cdot \mathbf{x}') \rho(\mathbf{x}')\, d^3x', \tag{6.58}
$$

and the vector potential becomes

$$
\mathbf{A}(\mathbf{x}) = -\frac{\mu_0 c k^2}{8\pi} \frac{e^{ikr}}{r} \left(1 - \frac{1}{ikr} \right) \int \mathbf{x}' (\mathbf{n} \cdot \mathbf{x}') \rho(\mathbf{x}')\, d^3x'. \tag{6.59}
$$

Although expressions for the electromagnetic fields valid over all of space could be evaluated using equations (6.8) and (6.9), we will, for the sake of simplicity, limit ourselves to the far (or radiation) fields. We then find

$$
\begin{aligned}
\mathbf{H} &= \frac{ik}{\mu_0} \mathbf{n} \times \mathbf{A} \\
\mathbf{E} &= \frac{ik}{\mu_0} Z_0 (\mathbf{n} \times \mathbf{A}) \times \mathbf{n}.
\end{aligned} \tag{6.60}
$$

Take note that in deriving equations (6.60) other terms arise (e.g., a term proportional to $(\mathbf{x}' \cdot \nabla)\mathbf{n}$; note that $\nabla \times \mathbf{n} = 0$), but they can safely be neglected in the far field since introduce an additional $1/r$ multiplying factor. Using equation (6.59) (keeping only the far field term), the magnetic field can be written as

$$\mathbf{H}(\mathbf{x}) = -\frac{\mu_0 c k^2}{8\pi}\frac{e^{ikr}}{r}\left[\mathbf{n}\times\int \mathbf{x}'(\mathbf{n}\cdot\mathbf{x}')\rho(\mathbf{x}')\,d^3x'\right]. \tag{6.61}$$

Since we expect the electric quadrupole term Q_{ij} (see equation (2.88)) to come out of this equation, we must somehow transform the term in bracket. More precisely, we consider one component of the corresponding vector

$$\left[\mathbf{n}\times\int \mathbf{x}(\mathbf{n}\cdot\mathbf{x})\rho(\mathbf{x})\,d^3x\right]_i = \varepsilon_{ijk} n_j \int x_k n_m x_m \rho(\mathbf{x})\,d^3x, \tag{6.62}$$

but since

$$Q_{km} = \int\left(3x_k x_m - \delta_{km} r^2\right)\rho(\mathbf{x})\,d^3x, \tag{6.63}$$

we have free to subtract the following term

$$\frac{1}{3}\varepsilon_{ijk}\delta_{jk}\int r^2\ d^3x = 0 \tag{6.64}$$

to equation (6.62) (note that $\varepsilon_{ijk}\delta_{jk} = 0$). Then equation (6.62) becomes

$$\begin{aligned}
\left[\mathbf{n}\times\int \mathbf{x}(\mathbf{n}\cdot\mathbf{x})\rho(\mathbf{x})\,d^3x\right]_i &= \varepsilon_{ijk}\int\left(n_j x_k n_m x_m - \frac{1}{3}\delta_{jk} r^2\right)\rho(\mathbf{x})\,d^3x \\
&= \varepsilon_{ijk}\int\left(n_j x_k n_m x_m - \frac{1}{3} n_j n_m \delta_{km} r^2\right)\rho(\mathbf{x})\,d^3x \\
&= \varepsilon_{ijk} n_j \int n_m \left(x_k x_m - \frac{1}{3}\delta_{km} r^2\right)\rho(\mathbf{x})\,d^3x \\
&= \frac{1}{3}\varepsilon_{ijk} n_j \int n_m Q_{km}\ d^3x.
\end{aligned} \tag{6.65}$$

Further defining a vector $\mathbf{Q}(\mathbf{n})$ such that its components are

$$Q_k = Q_{km} n_m, \tag{6.66}$$

then we have for the electromagnetic fields in the far field

$$\boxed{\begin{aligned}\mathbf{H} &= -\frac{ick^3}{24\pi}\frac{e^{ikr}}{r}\mathbf{n}\times\mathbf{Q}(\mathbf{n})\\ \mathbf{E} &= -\frac{Z_0 ick^3}{24\pi}\frac{e^{ikr}}{r}\left[\mathbf{n}\times\mathbf{Q}(\mathbf{n})\right]\times\mathbf{n}\end{aligned}} \tag{6.67}$$

Using equation (6.28) for the definition of the time-averaged power radiated per unit solid angle, and similar calculations as were used for the case of the electric dipole moment, we find that

$$\boxed{\frac{dP}{d\Omega} = \frac{c^2 Z_0 k^6}{1152\pi^2}\left|\left[\mathbf{n}\times\mathbf{Q}(\mathbf{n})\right]\times\mathbf{n}\right|^2} \tag{6.68}$$

for the electric quadrupole field. To evaluate the total power radiate, we first expand

$$\begin{aligned}\left|\left[\mathbf{n}\times\mathbf{Q}(\mathbf{n})\right]\times\mathbf{n}\right|^2 &= \mathbf{Q}^*\cdot\mathbf{Q} - \left|\mathbf{n}\cdot\mathbf{Q}\right|^2\\ &= Q_{ij}^* Q_{ik} n_j n_k - Q_{ij}^* Q_{km} n_i n_j n_k n_m .\end{aligned} \tag{6.69}$$

We must first evaluate the integrals $\int n_j n_k \, d\Omega$. We first note that since

$$\mathbf{n} = \mathbf{e}_x \sin(\theta)\cos(\varphi) + \mathbf{e}_y \sin(\theta)\sin(\varphi) + \mathbf{e}_z \cos(\theta), \tag{6.70}$$

and

$$\int\left[\sin(\theta)\sin(\varphi)\right]^2 d\Omega = \int\left[\sin(\theta)\cos(\varphi)\right]^2 d\Omega = \int \cos^2(\theta)\, d\Omega = \frac{4\pi}{3}, \tag{6.71}$$

while the integral of any other product of the components on $\mathbf{n}$ are zero, then

$$\int n_j n_k \, d\Omega = \frac{4\pi}{3}\delta_{jk}. \tag{6.72}$$

We also need to evaluate $\int n_i n_j n_k n_m \, d\Omega$. In this case, it is clear that integrals of this type will only be non-zero if they consist of a product of pairs. More precisely, if the integrand consists of the square of a given pair, then

$$\int\left[\sin(\theta)\sin(\varphi)\right]^4 d\Omega = \int\left[\sin(\theta)\cos(\varphi)\right]^4 d\Omega = \int \cos^4(\theta)\, d\Omega = \frac{4\pi}{5}, \tag{6.73}$$

while for the integrals made of the product of two distinct pairs we have

$$\begin{aligned}\int \left[\sin(\theta)\sin(\varphi)\right]^2 \left[\sin(\theta)\cos(\varphi)\right]^2 \, d\Omega \\ &= \int \left[\sin(\theta)\sin(\varphi)\right]^2 \cos^2(\theta)\, d\Omega \\ &= \int \left[\sin(\theta)\cos(\varphi)\right]^2 \cos^2(\theta)\, d\Omega = \frac{4\pi}{15}.\end{aligned} \tag{6.74}$$

Again, any other integral vanishes. We can combine these three different results into single equation

$$\int n_i n_j n_k n_m \, d\Omega = \frac{4\pi}{15}\left(\delta_{ij}\delta_{km} + \delta_{ik}\delta_{jm} + \delta_{im}\delta_{jk}\right). \tag{6.75}$$

We are now in a position to integrate equation (6.69), with the result that

$$\int \left|\left[\mathbf{n}\times\mathbf{Q}(\mathbf{n})\right]\times\mathbf{n}\right|^2 \, d\Omega = \frac{4\pi}{3}\left[Q_{ij}^* Q_{ij} - \frac{1}{5}\left(Q_{ii}^* Q_{jj} + 2Q_{ij}^* Q_{ij}\right)\right], \tag{6.76}$$

where we used $Q_{ij} = Q_{ji}$. Also, since we have from equation (6.63) that $Q_{ii} = 0$, then

$$\int \left|\left[\mathbf{n}\times\mathbf{Q}(\mathbf{n})\right]\times\mathbf{n}\right|^2 \, d\Omega = \frac{4\pi}{5} Q_{ij}^* Q_{ij}, \tag{6.77}$$

and

$$\boxed{P = \frac{c^2 Z_0 k^6}{1440\pi} Q_{ij}^* Q_{ij}} \tag{6.78}$$